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Twelve Erroneous Proofs

TOH Pee Choon
National Institute of Education
Nanyang Technological University, Singapore

Assessing how well students understand proofs is difficult. One way to achieve this is to present an erroneous proof and require students to identify the error. Twelve erroneous proofs on various topics in mathematics at the Secondary school level are presented as examples.

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The Importance of Proofs

The importance of proofs in the learning of mathematics cannot be overstated. The Ministry of Education of Singapore (2019, p. 41) listed the ability to "justify mathematical statements" and to "write mathematical arguments and proofs" as two of twelve assessment objectives in the national Secondary Mathematics Syllabus. These assessment objectives were meant to reflect the emphasis of the syllabus and described what students should know from learning the content. Similarly, the National Council of Teachers of Mathematics of the USA (2000, p. 56) advocated that "By the end of secondary school, students should be able to understand and produce mathematical proofs—arguments consisting of logically rigorous deductions of conclusions from hypotheses—and should appreciate the value of such arguments." The NCTM recommended four standards for *Reasoning & Proof*, one of which was to enable students to "develop and *evaluate* [emphasis added] mathematics arguments and proofs."

However, research in mathematics education has shown that not only students at the Secondary level (Stylianou et al., 2009) but also preservice teachers and undergraduates (Selden & Selden, 2003; Stylianou et al., 2015) face difficulties with writing and evaluating proofs. Selden and Selden (2003) presented four student-generated arguments—only one of which is a rigorous proof—as proofs of a single theorem to eight mathematics and secondary education mathematics majors. They found that the undergraduates had limited ability to determine whether these arguments count as proofs. For example, they noted certain errors in the arguments and thought these were valid and important, or they accepted as proof an argument that established the *converse*, rather than the actual theorem. Stylianou et al. (2015) collected data from 535 undergraduates from six universities through a 38-item multiple-choice test. The undergraduates were presented four different theorems, each accompanied by four "proofs". Correspondingly, sixteen of the items asked the undergraduates to rate each "proof" as either (a) logically flawed; (b) correct but not rigorous; (c) only true for some instances; or (d) a strong mathematical argument. The study classified 46% of the undergraduates as middle-performing (scoring 14-25 out of 38) and another 46% as low-performing (scoring 1-13). Additional self-reported data from some of these undergraduates' prior classroom experience suggested they had limited opportunities for seeing or constructing proofs. Furthermore, their involvement in these rare occasions were often as passive observers.

The humble aim of this article is to encourage Secondary school teachers to provide more opportunities for their students to be active participants in evaluating mathematics arguments and proofs. It is generally easier to understand a proof than to construct one. However, it is not so easy to assess how well a student has understood a proof. One possible way is to present an erroneous proof and require students to identify which part of the proof contains an error. The error should not be typical slips or mistakes committed by students like arithmetic mistakes, dropping a negative sign during algebraic manipulation, or forgetting to add a constant when performing integration. Ideally, the error should arise from misconceptions and when suitably identified would serve to reinforce learning. The task can be made more challenging by introducing uncertainty (Zaslavsky, 2005) through the occasional inclusion of error free proofs in the mix.

A Collection of Twelve Erroneous Proofs

The rest of this article contains a collection of twelve erroneous proofs curated for the Secondary classroom. These may not be immediately suitable for use in classrooms and teachers are invited to adapt these or craft their own erroneous proofs. There is no claim to originality, with the exception of the sixth, seventh and eighth which were modified from work of my previous students. Many of these erroneous proofs, sometimes called mathematical fallacies, are well known and can be found in the classical work by Ball and Coxeter (1987), and other more recent sources like Barbeau (2000). Variants can also be found readily online. My modest contribution lies in the presentation and also supplying an explanation which is otherwise absent in Ball and Coxeter (1987). For ease of reading, these erroneous proofs are bookended by the label ***EPx*** and the symbol ■. Readers are strongly encouraged to find out what is wrong with the presented proof before moving on to read the ensuing discussion. We begin with a well-known mathematical fallacy that has appeared in print for at least a hundred years, specifically the first edition of Ball and Coxeter (1987, p. 41) published in 1892.

EP1: "2 = 1"

Let $a = b$. Then we have $a^2 = ab$. Adding a^2 to both sides yields

$$a^2 + a^2 = a^2 + ab.$$

Now subtract $2ab$ from both sides.

$$2a^2 - 2ab = a^2 - ab.$$

Cancelling the common factor $a(a - b)$, we have

$$2 = 1. \quad ■$$

There are many different variants of ***EP1***, all of which involve the crucial mistake of cancelling a common factor which equals 0. Instead of simply telling students *not to divide by zero*, it would probably be more instructive to show them that unexpected contradictions may arise. It is also a good opportunity to reinforce the idea that division by a factor x, should be viewed as multiplication with the multiplicative inverse of x. The property that "*0 multiplied to any other real number equals 0*" prohibits the existence of a multiplicative inverse for 0. Otherwise, it leads to calculations like the following.

EP2: "All numbers are equal"
We have
$$0 \times 9999 = 0 = 0 \times 2.$$

Let 0^{-1} denote the multiplicative inverse of 0, i.e. $0^{-1} \times 0 = 1$. Then

$$\begin{aligned} 0^{-1} \times 0 \times 9999 &= 0^{-1} \times 0 \times 2 \\ \Longrightarrow 1 \times 9999 &= 1 \times 2 \\ \Longrightarrow \quad 9999 &= 2. \end{aligned}$$

As the choices of 9999 and 2 were essentially arbitrary, this means that we can use the same method to show that all numbers are equal. ■

EP3: "Absurd surds"
We know that $\sqrt{a}\sqrt{b} = \sqrt{ab}$, thus

$$\left(\sqrt{-1}\right)^2 = \sqrt{-1}\sqrt{-1} = \sqrt{-1 \times (-1)} = \sqrt{1}.$$

In other words,
$$1 = \sqrt{1} = \left(\sqrt{-1}\right)^2 = -1. \blacksquare$$

Recall that for real numbers, $\sqrt{x}$ is only defined for positive x. Hence, the algebraic relation $\sqrt{a}\sqrt{b} = \sqrt{ab}$ is only valid for positive values of a and b, a good reminder for students to check for conditions where mathematical results hold. The reminder remains equally sound even when extended to complex numbers. In such cases, while $\sqrt{-1}$ is well defined, the relation $\sqrt{a}\sqrt{b} = \sqrt{ab}$ no longer holds for all a and b. Next, we have an analogous example involving logarithms and exponentials.

EP4: "Ludicrous logarithms"
We know that $\log_e a + \log_e b = \log_e(ab)$, thus

$$2\log_e(-1) = \log_e(-1) + \log_e(-1) = \log_e(-1 \times (-1)) = \log_e 1 = 0.$$

This means that $\log_e(-1)$ must take the value of 0. Setting this value in the exponential function gives
$$-1 = e^{\log_e(-1)} = e^0 = 1. \blacksquare$$

The logarithm $\log_e x$ is only defined for positive x, unless we are working with complex numbers. Hence, the addition rule should only be used for positive values of a and b. Instead of using the addition rule, we could also end up with the same error by using

$$2\log_e a = \log_e(a^2).$$

Alternatively, we can use ***EP4*** to demonstrate how contradictions may arise if we *simplistically* extend the domain of the logarithm function to negative numbers.

EP5: "More absurd surds"
Let $x = 1 + \frac{1}{\sqrt{2}}$. We simplify the expression to

$$x = \frac{\sqrt{2}+1}{\sqrt{2}} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{1}{2-\sqrt{2}}.$$

Clearing denominators gives

$$\begin{aligned}(2-\sqrt{2})x &= 1\\ \Rightarrow -\sqrt{2}x &= 1-2x.\end{aligned}$$

We double the equation and add $2 + x^2$,

$$2 - 2\sqrt{2}x + x^2 = 4 - 4x + x^2.$$

Both sides are now "squares", i.e.

$$\left(\sqrt{2} - x\right)^2 = (2-x)^2.$$

Taking square root gives,

$$\begin{aligned}\sqrt{2} - x &= 2 - x\\ \Rightarrow \sqrt{2} &= 2. \;\blacksquare\end{aligned}$$

The problem in ***EP5*** results from the failure to understand the distinction between the quadratic equation $x^2 = a$, and its solutions $x = \pm\sqrt{a}$. Students may also not understand what is meant by the $\pm$ symbol. The penultimate step should read

$$\sqrt{2} - x = \pm(2-x),$$

which means

$$\sqrt{2} - x = 2 - x \quad \text{or} \quad \sqrt{2} - x = -(2-x).$$

There are now two cases. The first leads to a contradiction as previously presented, while the second leads to $x = \frac{2+\sqrt{2}}{2} = 1 + \frac{1}{\sqrt{2}}$, our original starting point. Many students learn to solve quadratic equations procedurally. (See also ***EP6***.) Thus, an alternative way to present ***EP5*** could have the following final steps.

$$\begin{aligned}\left(\sqrt{2} - x\right)^2 &= (2-x)^2\\ \Rightarrow \pm\sqrt{\left(\sqrt{2} - x\right)^2} &= \pm\sqrt{(2-x)^2}\\ \Rightarrow \sqrt{2} - x &= 2 - x\\ \Rightarrow \sqrt{2} &= 2.\end{aligned}$$

As in ***EP2***, the choice of numbers is arbitrary although some effort may be necessary to make the calculation work. I have seen variants showing "$4 = 5$" and also a recent example purporting "$\pi = 3$" in a blog post by Steckles (2021). I chose to include surds to

provide more opportunities to reinforce algebraic manipulation skills, and also to make it less obvious that x was just the mean of 2 and $\sqrt{2}$.

EP6: "Quadratic quandary"

We prove that the quadratic equation $x^2 - 5x + 4 = 0$ has the solutions $x = 1$ or $x = 4$.

Proof: Adding the two solutions give $2x = 5$, which may then be added to the original equation to get

$$\begin{aligned} x^2 - 5x + 4 + 2x &= 0 + 5 \\ \Rightarrow\ x^2 - 3x &= 5 - 4 \\ \Rightarrow x(x-3) &= 1 \\ \Rightarrow\ x = 1 \ \text{ or } \ x - 3 &= 1 \\ \Rightarrow\ x = 1 \ \text{ or } \ x &= 4.\ \blacksquare \end{aligned}$$

Thus far, all the five previous erroneous proofs yielded obviously false conclusions. In ***EP6***, we have an erroneous proof of a correct result: the solutions for the equation are indeed $x = 1$ or $x = 4$. I deliberately introduced two errors in order to arrive at the correct answer.

The first error is that the solution x cannot be equal to both 1 and 4 *at the same time*. This point may not be very clear to students and is further exacerbated by fact that it is perfectly alright to say "the quadratic polynomial $x^2 - 5x + 4$ has exactly two roots, namely, 1 and 4."

The second error stems from students learning to solve quadratic equations procedurally. The teacher demonstrates that

$$\begin{aligned} x^2 - 5x + 4 &= 0 \\ \Rightarrow (x-1)(x-4) &= 0 \\ \Rightarrow\ x - 1 = 0 \ \text{ or } \ x - 4 &= 0 \\ \Rightarrow\ x = 1 \ \text{ or } \ x &= 4. \end{aligned}$$

The student over-generalizes and assume that the same procedure can be applied to $x(x-3) = 1$. Discussing this erroneous proof can provide a springboard for the teacher to explain that the original solution is only valid due to what is called the *zero-product property*, i.e.

If $ab = 0$, then $a = 0$ or $b = 0$.

EP7: "Irrational contradiction"

We prove that the sum of two rational numbers is rational.

Proof: Suppose to the contrary that the sum of two rational numbers x and y is irrational. We write $x = \frac{a}{b}$ and $y = \frac{c}{d}$, where a, b, c, d are integers with $b \neq 0, d \neq 0$. Then

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

Since both $ad + bc$ and bd are integers, and we have $bd \neq 0$, we can conclude that $x + y$ is rational. This contradicts the assumption that the sum was irrational. ■

The above is a typical example of solutions that I receive fairly often from my students. There is nothing wrong with the proof, other than the fact that it is not a proof by contradiction. The proof can and should be converted to a direct proof (Pólya, 1957, p. 162-171).

EP8: "Odd pair"

We prove that the sum of two odd numbers is even.

Proof: Suppose to the contrary that the sum of two odd numbers is odd. If x is an odd number, then $x + x = 2x$, is clearly even. This contradicts the assumption that the sum was odd. ■

The above erroneous proof is modified from the work of a previous student. The actual result that he was trying to prove involved a more advanced concept in number theory. I kept the structure of his argument but changed it to the above context to help him see what was wrong with his reasoning. Like ***EP7***, the above proof is not really a proof by contradiction. When written as a direct proof, it only shows that twice of an odd number is even and not the sum of two arbitrary odd numbers is even as was required.

EP9: "Factoring sin"

We prove the trigonometric identity

$$(\sin A)^2 - (\sin B)^2 = \sin(A+B)\sin(A-B).$$

Proof: Consider the left side,

$$\begin{aligned}(\sin A)^2 - (\sin B)^2 &= [\sin(A) - \sin(B)]\,[\sin(A) + \sin(B)] \\ &= [\sin(A-B)]\,[\sin(A+B)] \\ &= \sin(A+B)\sin(A-B).\end{aligned}$$

Since we have shown that the left side is equal to the right side, the proof is complete. ■

This erroneous proof is modified from Barbeau (2000, p. 33) and reflects a common misconception of students who do not have a strong understanding of functions and the functional notation. The identity is true and can be proved using the addition formula.

$$\begin{aligned}\sin(A+B)\sin(A-B) &= [\sin A\cos B + \cos A\sin B][\sin A\cos B - \cos A\sin B] \\ &= (\sin A\cos B)^2 - (\cos A\sin B)^2 \\ &= (\sin A)^2(1 - (\sin B)^2) - (1 - (\sin A)^2)(\sin B)^2 \\ &= (\sin A)^2 - (\sin B)^2.\end{aligned}$$

The final three erroneous proofs come from geometry and are particularly ingenious. They are originally due to Ball (Ball & Coxeter, p. 77-81.) In particular, ***EP11*** which claims that *all triangles are isosceles* is quite well known and often misattributed to Charles Dodgson, better known as Lewis Carroll, the author of *Alice in Wonderland*. This despite Ball publishing the following footnote in his book (no later than 1914, in the sixth edition).

> I believe that this and the fourth of these fallacies were first published in this book. They particularly interested Mr. C.L. Dodgson; see the *Lewis Carroll Picture Book*, London, 1899, pp. 264, 266, where they appear in the form in which I originally gave them. (p. 77)

To further clarify Ball's footnote: "this" refers to ***EP10***, and I have separated the two parts of "the fourth" into ***EP11*** and ***EP12***.

EP10: "Ridiculous right angles"
Let $ABCD$ be a rectangle and let E be a point outside the rectangle such that the segment $AE = AB = DC$. Since CB and CE are not parallel, their respective perpendicular bisectors will meet at a point, say F. Construct the segments FA, FE, FC and FD. Finally, let G be the midpoint of CE, and H be the midpoint of DA. (See Figure 1.)

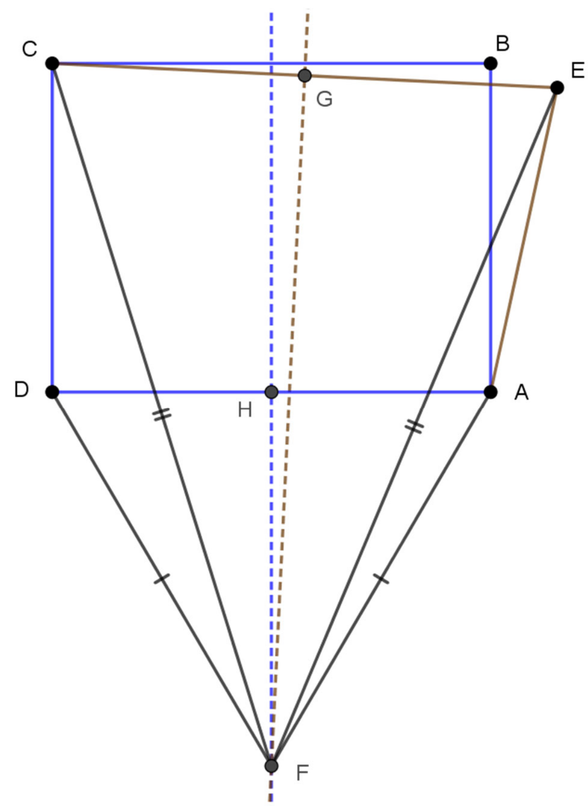

Figure 1. EP10: Ridiculous right angles

Since $DH = HA$ and HF is perpendicular to CB and hence also to DA, we can conclude that $\triangle FHD$ and $\triangle FHA$ are congruent by SAS congruency test. Consequently $FA = FD$ and $\angle FDH = \angle FAH$. An analogous argument shows that $\triangle FGC$ and $\triangle FGE$ are also congruent and thus $FE = FC$. Since $AE = CD$, we can now use SSS congruency test to conclude that $\triangle FCD$ and $\triangle FEA$ are congruent. This would necessarily mean that $\angle FDC = \angle FAE$. Thus

$$\angle DAE = \angle FAE - \angle FAH = \angle FDC - \angle FDH = \angle ADC = 90^\circ. \blacksquare$$

Most students would be able to identify that the conclusion $\angle DAE = 90^\circ$ is wrong but they would probably not be able to point out the error in the "proof". It turns out that FE does not intersect the rectangle, hence it is not true that $\angle FAE = \angle DAE + \angle FAH$. In Figure 2, the locus of E, which is a circle centred at A with radius AB, is constructed to clarify the situation. It is

particularly instructive to demonstrate this with a dynamic geometry software. As E approaches B along the locus, the intersection F moves further away. In the limit when $E = B$, the two perpendicular bisectors coincide and there is no unique point of intersection.

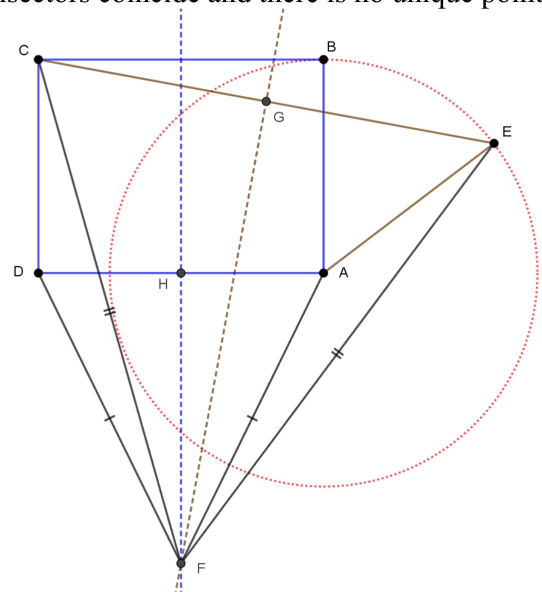

Figure 2. $\triangle FCD$ and $\triangle FEA$ are indeed congruent

EP11: "All triangles are isosceles"

Let ABC be an arbitrary triangle and let D be the midpoint of BC. Construct the perpendicular bisector of BC and let P be the point of intersection of this perpendicular bisector with the angle bisector of $\angle CAB$. From P, drop perpendicular points E and F on CA and AB respectively. (See Figure 3.)

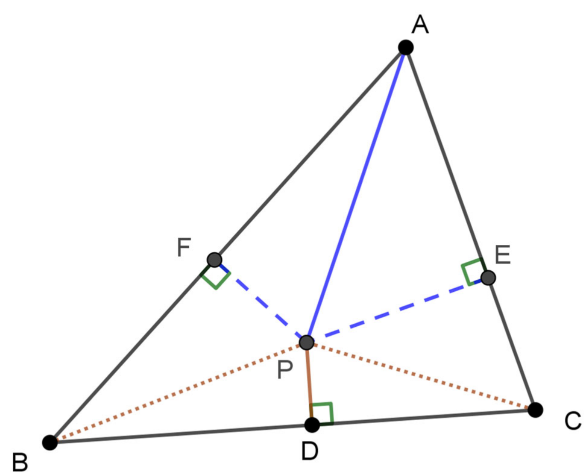

Figure 3. EP11: All triangles are isosceles

Since AP is the angle bisector, by AAS congruency test, $\triangle APE$ and $\triangle APF$ are congruent. Hence $AE = AF$ and $PE = PF$. It is also true that $PB = PC$ since $\triangle DPB$ and $\triangle DPC$ are

congruent. We can now use the RHS congruency test to conclude that $\triangle PEC$ and $\triangle PFB$ are congruent. This means that $EC = FB$. Together with $AE = AF$, we see that $AC = AB$ and our arbitrary triangle is in fact isosceles! ■

Analogous to ***EP10***, the crucial mistake in the proof was the assumption that the point of intersection P lies within the triangle. If $\triangle ABC$ was isosceles, the angle bisector and perpendicular bisector would coincide and there is no unique point of intersection. However, if the triangle was not isosceles, the intersection point must lie outside of the triangle. Otherwise, the argument of ***EP11*** could be used to prove that the triangle was isosceles, giving rise to a contradiction. Let us now try to understand the consequence of point P being outside of the triangle.

EP12: "All triangles are still isosceles"

Let ABC be an arbitrary triangle and let D be the midpoint of BC. Construct the perpendicular bisector of BC and let P be the point of intersection of this perpendicular bisector with the angle bisector of $\angle CAB$. From the previous discussion in ***EP11***, P must lie outside of $\triangle ABC$. From P, drop perpendicular points E and F on AC and AB produced respectively. (See Figure 4.)

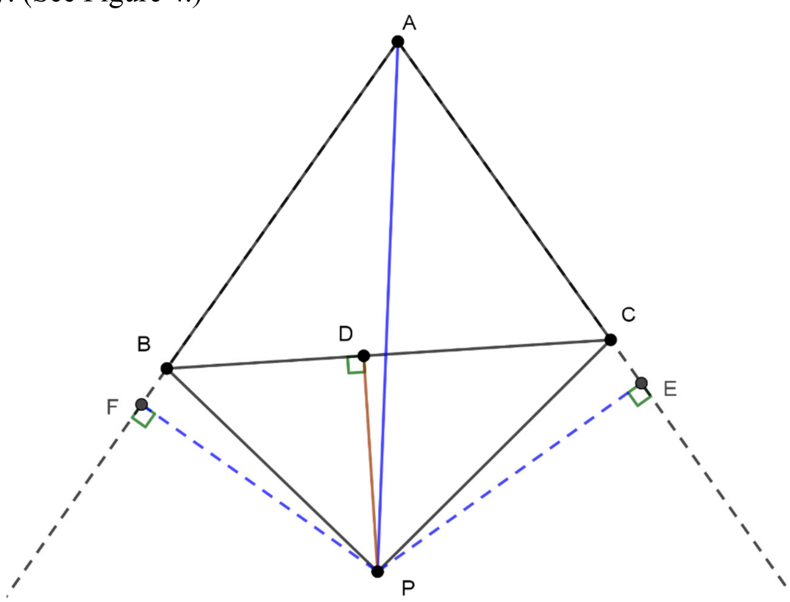

Figure 4. EP12: All triangles are still isosceles

In this case, $\triangle APE$ and $\triangle APF$ are still congruent by the AAS congruency test. Hence, we have $AE = AF$ and $PE = PF$. It is still true that $PC = PB$ since $\triangle DPC$ and $\triangle DPB$ are congruent. We can again use the RHS congruency test to conclude that $\triangle PEC$ and $\triangle PFB$ are congruent. This means that $CE = BF$. Thus

$$AC = AE - CE = AF - BF = AB.$$

Our arbitrary triangle remains isosceles! ■

Again, this would seem like a perfectly rigorous proof to most students until hopefully some of them begin to question the positions of E and F, the foots of the perpendicular from P. The fact is that one of the two will always lie within the triangle while the other will be outside (on

one of the sides produced.) Regardless of their positions, $AE = AF$ and $CE = BF$ continue to hold. Thus, if F lies on AB, then $AB = AF + BF$, while $AC = AE - CE = AF - BF$, and so no contradictions arise.

Conclusion

Twelve erroneous proofs on Secondary mathematics have been presented. The author hopes that teachers, who agree that such proofs can be a useful pedagogical tool, can adapt, improve and craft other erroneous proofs for their own classrooms.

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Author

TOH, Pee Choon (peechoon.toh@nie.edu.sg),
National Institute of Education, Nanyang Technological University, Singapore,
1 Nanyang Walk, Singapore 637616